Differential inequalities for Riesz means and Weyl-type bounds for eigenvalues [1]

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Abstract

We derive differential inequalities and difference inequalities for Riesz means of eigenvalues of the Dirichlet Laplacian,

$$R_\sigma(z) := \sum_k (z - \lambda_k)_+^\sigma.$$

Here $\{\lambda_k\}_{k=1}^\infty$ are the ordered eigenvalues of the Laplacian on a bounded domain $\Omega \subset \mathbb{R}^d$, and $x_+ := \max(0, x)$ denotes the positive part of the quantity x. As corollaries of these inequalities, we derive Weyl-type bounds on λ_k, on averages such as $\overline{\lambda_k} := \frac{1}{k} \sum_{\ell \le k} \lambda_\ell$, and on the eigenvalue counting function. For example, we prove that for all domains and all $k \ge j \frac{1 + \frac{d}{2}}{1 + \frac{d}{4}}$,

$$\overline{\lambda_k}/\overline{\lambda_j} \le 2 \left(\frac{1 + \frac{d}{4}}{1 + \frac{d}{2}} \right)^{1 + \frac{2}{d}} \left(\frac{k}{j} \right)^{\frac{2}{d}}.$$

Key words: Laplacian, Weyl law, Dirichlet problem, Riesz means, universal bounds
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1 Introduction

This article is concerned with the spectrum of the Laplace operator $-\Delta$ on a bounded Euclidean domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, with vanishing Dirichlet boundary conditions on $\partial\Omega$, *i.e.*, the classic problem of the vibratory modes of a finite elastic body with fixed boundary. With straightforward changes it is also possible to treat Schrödinger operators $-\Delta + V(x)$ with discrete spectra. Indeed, virtually no changes at all are necessary when $V(x) \geq 0$, other than possibly the upper bound in (2.11). For simplicity of exposition, however, we refer below only to the Laplacian.

The spectrum $\{\lambda_k\}_{k=1}^\infty$, of $-\Delta$ is subject to the Weyl law, [36,15,10,32]

$$\lambda_k \sim 4\pi \frac{\left(\Gamma(1 + \frac{d}{2})\right)^{\frac{2}{d}}}{|\Omega|^{\frac{2}{d}}} k^{\frac{2}{d}} \tag{1.1}$$

as $k \to \infty$; to the Berezin-Li-Yau inequality, [11,31,26,29,28]

$$\overline{\lambda_k} := \frac{1}{k} \sum_{\ell \leq k} \lambda_\ell \geq 4\pi \frac{\left(\Gamma(1 + \frac{d}{2})\right)^{\frac{2}{d}}}{|\Omega|^{\frac{2}{d}}} \frac{k^{\frac{2}{d}}}{1 + \frac{2}{d}}; \tag{1.2}$$

and to other familiar constraints relating the spectrum to the geometry of Ω. The Berezin-Li-Yau inequality is a notable example of an inequality for which both sides are of the same order in k, when the asymptotic expression in (1.1) is substituted for λ_k. A formula or expression where this is the case is said to be of *Weyl-type*.

At the same time, the spectrum is subject to "universal bounds" by which certain expressions involving eigenvalues dominate others with no reference to the geometry of Ω. The expressions occurring in known universal bounds include moments of eigenvalues, in particular sums $\sum_{j=1}^k \lambda_j$ as in Berezin-Li-Yau. (See [3,22] for a review of universal spectral bounds.) Although early universal bounds like that of Payne, Pólya, and Weinberger [33] were not of Weyl-type, universal bounds of Weyl-type have been known since H. C. Yang's unpublished 1991 article [37,21,30,3,7,13,14,8,9,22].

One of the goals here is to show that trace identities of the type introduced in [21] imply tight, Weyl-type bounds on ratios of eigenvalues belonging to Laplace spectra. To our knowledge this has not been much explored except by Hermi [22] and by Cheng and Yang in [14], with some remarks in [13]. Analogous bounds for Schrödinger operators in one dimension can be found in the work of Ashbaugh and Benguria [4]. Ratios involving averages $\overline{\lambda_k}$ will arise in a natural way through this analysis.

The earliest bound for the ratio of a large eigenvalue to the fundamental eigenvalue is due to Ashbaugh and Benguria [6], who proved

$$\frac{\lambda_{2^m}}{\lambda_1} \le \left(\frac{j_{d/2,1}^2}{j_{d/2-1,1}^2} \right)^m. \tag{1.3}$$

While optimal for low-lying eigenvalues, (1.3) is not of Weyl-type since the right side of the inequality behaves like $k^{5.77078/d}$ as $k \to \infty$ (see the details in [22]). Weyl-type bounds were proved in [22], of the form

$$\frac{\lambda_{k+1}}{\lambda_1} \le 1 + \left(1 + \frac{d}{2} \right)^{2/d} H_d^{2/d} k^{2/d}, \tag{1.4}$$

and

$$\frac{\overline{\lambda}_k}{\lambda_1} \le 1 + \frac{H_d^{2/d}}{1 + \frac{2}{d}} k^{2/d}, \tag{1.5}$$

where

$$H_d = \frac{2\,d}{j_{d/2-1,1}^2 J_{d/2}^2(j_{d/2-1,1})}. \tag{1.6}$$

As usual, $j_{\alpha,p}$ denotes the p-th positive zero of the Bessel function $J_\alpha(x)$ [1]. In [14] Cheng and Yang prove that

$$\frac{\lambda_{k+1}}{\lambda_1} \le \left(1 + \frac{4}{d} \right) k^{\frac{2}{d}}, \tag{1.7}$$

as well as some incremental improvements for large values of k, d of the form

$$\frac{\lambda_{k+1}}{\lambda_1} \le C_0(d,k) k^{\frac{2}{d}}. \tag{1.8}$$

For instance, for $k \ge d + 1$, Cheng and Yang [14] improve (1.7) to

$$\frac{\lambda_{k+1}}{\lambda_1} \le \left(1 + \frac{4}{d} \right) \left(1 + \frac{8}{d+1} + \frac{8}{(d+1)^2} \right)^{\frac{1}{2}} (d+1)^{-\frac{2}{d}} k^{\frac{2}{d}}. \tag{1.9}$$

For comparison, by combining the Weyl law with the Rayleigh-Faber-Krahn inequality [18,24,25], there immediately results an asymptotic upper bound on λ_k/λ_1, of the form

$$\frac{4\pi}{\lambda_1^*} \frac{\left(\Gamma(1 + \frac{d}{2}) \right)^{\frac{2}{d}}}{|\Omega|^{\frac{2}{d}}} k^{\frac{2}{d}} = \frac{4 \left(\Gamma(1 + \frac{d}{2}) \right)^{\frac{4}{d}}}{j_{\frac{d}{2}-1,1}^2} k^{\frac{2}{d}}, \tag{1.10}$$

where

$$\lambda_1^* = \frac{\pi\, j_{\frac{d}{2}-1,1}^2}{\left(\Gamma(1 + \frac{d}{2})\, |\Omega| \right)^{\frac{2}{d}}}$$

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is the explicit value of the fundamental eigenvalue when Ω is a ball.

Ideally, a Weyl-type bound would contain a constant commensurate with that on the right side of (1.10). The bound of (1.7) is numerically nearly 4.34 times as large as the ideal when $d = 2$. It will be shown below that the constants in (1.7) and (1.9) can be reduced. (See Table 3 below.)

Our technique will make use of differential inequalities and difference inequalities for Riesz means of eigenvalues. Safarov, Laptev, and Weidl have long advocated Riesz means as a tool for understanding inequalities like those of Lieb-Thirring and Berezin-Li-Yau, and we draw some of our inspiration from [34,35,26,27,28,29].

2 Differential and difference inequalities for Riesz means of eigenvalues

For background information we refer to the monograph of Chandrasekharan and Minakshisundaram [12], where Riesz means are referred to as *typical means*. Recall that if $\{\lambda_k\}_{k=1}^\infty$ is an increasing sequence of real numbers, then for any real z, the *Riesz mean of order σ* of $\{\lambda_k\}$ can be defined as

$$R_\sigma(z) := \sum_k (z - \lambda_k)_+^\sigma \tag{2.1}$$

for $\sigma > 0$. When $\sigma = 0$, we interpret $R_0(z)$ as the eigenvalue-counting function $R_0(z) := \mathcal{N}(z) := \lim_{\sigma \downarrow 0} R_\sigma(z)$.

One of the main results consists of differential inequalities for $R_\sigma(z)$ with respect to z and difference inequalities for $R_\sigma(z)$ with respect to σ.

Theorem 2.1 *For $0 < \sigma \le 2$ and $z \ge \lambda_1$,*

$$R_{\sigma-1}(z) \ge \left(1 + \frac{d}{4}\right)\frac{1}{z}R_\sigma(z); \tag{2.2}$$

$$R'_\sigma(z) \ge \left(1 + \frac{d}{4}\right)\frac{\sigma}{z}R_{\sigma(z)}; \tag{2.3}$$

and consequently

$$\frac{R_\sigma(z)}{z^{\sigma+\frac{d\sigma}{4}}}$$

is a nondecreasing function of z.

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For $2 \leq \sigma < \infty$ and $z \geq \lambda_1$,

$$R_{\sigma-1}(z) \geq \left(1 + \frac{d}{2\sigma}\right)\frac{1}{z}R_\sigma(z); \tag{2.4}$$

$$R_\sigma'(z) \geq \left(\sigma + \frac{d}{2}\right)\frac{1}{z}R_\sigma(z); \tag{2.5}$$

and consequently

$$\frac{R_\sigma(z)}{z^{\sigma+\frac{d}{2}}}$$

is a nondecreasing function of z.

Remarks 2.2

1. The values of z are restricted for the simple reason that $R_\sigma(z) = 0$ for $\sigma > 0, z \leq \lambda_1$, and is undefined for $\sigma \leq 0, z \leq \lambda_1$.

2. It is only necessary to prove (2.2) and (2.4), since from (2.1),

$$R_\sigma'(z) = \sigma R_{\sigma-1}(z),$$

and since solving the elementary differential inequalities (2.3) and (2.5) easily yields the remaining statements about functions of the form $R_\sigma(z)/z^p$.

3. In the case $\sigma = 2$, (2.4) is related to the inequality of H. C. Yang [37,21,30,8,20].

4. Differential inequalities were considered in [21] for the partition function (or trace of the heat kernel) $Z(t) := \mathtt{tr}\left(e^{t\Delta}\right)$, where it was shown that $Z(t)t^{d/2}$ is a nonincreasing function. In a future article [20], it will be shown that the differential inequalities of Theorem 2.1 imply the equivalence of the Berezin-Li-Yau inequality and a classical inequality due to Kac [23] about the partition function.

To prepare the proof we state two lemmas.

Lemma 2.1 *Denoting the L^2-normalized eigenfunctions of the Laplace operator $\{u_j\}$, let*

$$T_{\alpha jm} := \left|\left(\frac{\partial u_j}{\partial x_\alpha}, u_m\right)\right|^2$$

for $j, m = 1, \dots$ and $\alpha = 1, \dots, d$. Then for each fixed α,

$$R_\sigma(z) = 2 \sum_{j,m:\lambda_j \neq \lambda_m} \frac{(z - \lambda_j)_+^\sigma - (z - \lambda_m)_+^\sigma}{\lambda_m - \lambda_j} T_{\alpha jm} + 4 \sum_{j,q:\lambda_j \leq z < \lambda_q} \frac{(z - \lambda_j)^\sigma}{\lambda_q - \lambda_j} T_{\alpha jq}.$$
$$\tag{2.6}$$

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This lemma is the trace identity of Harrell and Stubbe ([21] Theorem 1, (4)), specialized to $f(\lambda) = (z-\lambda)_+^\sigma$. Versions of the trace identity of [21] also appear in some later articles, e.g., [30,7,8,16].

Lemma 2.2 *Let $0 < x < y$ and $\sigma \geq 0$. Then*

$$\frac{y^\sigma - x^\sigma}{y - x} \leq C_\sigma \left(y^{\sigma-1} + x^{\sigma-1}\right), \tag{2.7}$$

where

$$C_\sigma := \begin{cases} \frac{\sigma}{2}, & \text{if } 0 \leq \sigma < 1 \\ 1, & \text{if } 1 \leq \sigma \leq 2 \\ \frac{\sigma}{2}, & \text{if } 2 \leq \sigma < \infty. \end{cases}$$

Proof. By a scaling, it suffices to assume $x = 1$. We then seek the supremum of

$$\frac{y^\sigma - 1}{(y-1)(y^{\sigma-1} + 1)}$$

for $1 < y < \infty$. A calculus exercise shows that the supremum is approached as $y \downarrow 1$ when $0 < \sigma < 1$ or $2 \leq \sigma < \infty$, whereas it is approached as $y \to \infty$ when $1 \leq \sigma \leq 2$. (It's constant when $\sigma = 2$.) The stated values are obtained with l'Hôpital's rule in the cases $0 < \sigma < 1$ and $2 \leq \sigma < \infty$. □

Proof of Theorem 2.1. Let the first term on the right of (2.6) be

$$G(\sigma, z, \alpha) := 2 \sum_{j,m:\lambda_j \neq \lambda_m} \frac{(z - \lambda_j)_+^\sigma - (z - \lambda_m)_+^\sigma}{\lambda_m - \lambda_j} T_{\alpha jm}.$$

By Lemma 2.2, this expression simplifies to

$$\begin{aligned} G(\sigma, z, \alpha) \quad &= 2 \sum_{j,m:\lambda_{j,m} \leq z, \lambda_j \neq \lambda_m} \frac{(z - \lambda_j)^\sigma - (z - \lambda_m)^\sigma}{(z - \lambda_j) - (z - \lambda_m)} T_{\alpha jm} \\ &\leq 2\, C_\sigma \sum_{j,m:\lambda_{j,m} \leq z} \left((z - \lambda_j)^{\sigma-1} + (z - \lambda_m)^{\sigma-1}\right) T_{\alpha jm} \\ &= 4\, C_\sigma \sum_{j,m:\lambda_{j,m} \leq z} (z - \lambda_j)_+^{\sigma-1} T_{\alpha jm} \end{aligned}$$

by symmetry in $j \leftrightarrow m$. Extending the sum to all m and subtracting the same quantity from the final term in (2.6), we find

$$R_\sigma(z) \leq 4C_\sigma \sum_{j:\lambda_j \leq z,\, \text{all } m} (z - \lambda_j)_+^{\sigma-1} T_{\alpha jm} + 4H(\sigma, z, \alpha), \tag{2.8}$$

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where

$$H(\sigma, z, \alpha) := \sum_{j,q:\lambda_j \leq z < \lambda_q} T_{\alpha jq} (z - \lambda_j)^{\sigma-1} \left(\frac{(z - \lambda_j) - C_\sigma (\lambda_q - \lambda_j)}{\lambda_q - \lambda_j} \right). \quad (2.9)$$

Next observe that because $\{u_m\}$ is a complete orthonormal set,

$$\sum_m T_{\alpha jm} = \left\| \frac{\partial u_j}{\partial x_\alpha} \right\|_2^2$$

and thus

$$\sum_{m,\alpha} T_{\alpha jm} = \|\nabla u_j\|_2^2 = \lambda_j.$$

Therefore we may average over $\alpha = 1, \ldots, d$ in (2.8) to obtain

$$R_\sigma(z) \leq \frac{4C_\sigma}{d} \sum_j (z - \lambda_j)_+^{\sigma-1} \lambda_j + \frac{4}{d} \sum_{\alpha=1}^d H(\sigma, z, \alpha),$$

or, since

$$\sum_j (z - \lambda_j)_+^{\sigma-1} \lambda_j = z R_{\sigma-1}(z) - R_\sigma(z),$$

$$\left(1 + \frac{4C_\sigma}{d} \right) R_\sigma - \frac{4zC_\sigma}{d} R_{\sigma-1}(z) \leq \frac{4}{d} \sum_{\alpha=1}^d H(\sigma, z, \alpha). \quad (2.10)$$

We consider three cases:

Case I. $1 \leq \sigma \leq 2$. In this case $C_\sigma = 1$ and $H(\sigma, z, \alpha) \leq 0$, establishing that

$$\left(1 + \frac{4}{d} \right) R_\sigma - \frac{4z}{d} R_{\sigma-1}(z) \leq 0,$$

which is equivalent to (2.2) for this range of σ.

Case II. $0 < \sigma < 1$.

Since the sum defining H runs over $\lambda_q > z$,

$$\frac{(z - \lambda_j) - C_\sigma (\lambda_q - \lambda_j)}{\lambda_q - \lambda_j} \leq \frac{(\lambda_q - \lambda_j) - C_\sigma (\lambda_q - \lambda_j)}{\lambda_q - \lambda_j} = 1 - C_\sigma.$$

Therefore

$$H(\sigma, z, \alpha) \leq (1 - C_\sigma) \sum_{j,q:\lambda_q \geq z} T_{\alpha jq} (z - \lambda_j)_+^{\sigma-1},$$

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and since in this case $1 - C_\sigma = 1 - \sigma/2 > 0$, we may extend the sum over all q, obtaining

$$\sum_{\alpha=1}^{d} H(\sigma, z, \alpha) \leq (1 - C_\sigma) \sum_j \lambda_j \left(z - \lambda_j \right)_+^{\sigma-1}.$$

Substituting this into (2.10), there is a cancellation of the C_σ's, and we again obtain

$$\left(1 + \frac{4}{d} \right) R_\sigma - \frac{4z}{d} R_{\sigma-1}(z) \leq 0,$$

equivalent to (2.2).

Case III. $\sigma > 2$. Arguing as in Case II, we come as far as

$$H(\sigma, z, \alpha) \leq (1 - C_\sigma) \sum_{j,q:\lambda_j \leq z < \lambda_q} T_{\alpha j q} \left(z - \lambda_j \right)^{\sigma-1},$$

but since now $(1 - C_\sigma) = 1 - \sigma/2 < 0$, we cannot extend the sum in q to simplify $T_{\alpha j q}$, and only conclude that

$$\left(1 + \frac{4C_\sigma}{d} \right) R_\sigma - \frac{4z C_\sigma}{d} R_{\sigma-1}(z) \leq 0,$$

which is (2.4).

$\square$

Convexity provides insight into why the proof divides into three ranges of σ. The Riesz mean $R_\sigma(z)$ loses convexity in z when $\sigma < 1$, and its derivative loses convexity already when $\sigma < 2$. Thus both $\sigma = 1$ and 2 are values at which necessary inequalities in the proof reverse.

Since by the theorem, $R_\sigma(z)z^{-p}$ is a nondecreasing function for appropriate values of p, we obtain lower bounds of the form $R_\sigma(z) \geq Cz^p$ for all $z \geq z_0$ as soon as $R_\sigma(z_0)$ is known, or estimated from below. Upper bounds can be obtained from the limiting behavior of $R_\sigma(z)$ as $z \to \infty$, as given by the Weyl law.

Corollary 2.3 *For all $\sigma \geq 2$ and $z \geq \left(1 + \frac{2\sigma}{d} \right) \lambda_1$,*

$$\left(\frac{2\sigma}{d} \right)^\sigma \lambda_1^{-\frac{d}{2}} \left(\frac{z}{1 + \frac{2\sigma}{d}} \right)^{\sigma + \frac{d}{2}} \leq R_\sigma(z) \leq L_{\sigma,d}^{cl} |\Omega| z^{\sigma + \frac{d}{2}}, \qquad (2.11)$$

where

$$L_{\sigma,d}^{cl} := \frac{\Gamma(\sigma + 1)}{(4\pi)^{\frac{d}{2}} \Gamma \left(\sigma + 1 + \frac{d}{2} \right)}. \qquad (2.12)$$

Remark 2.4 The upper bound on $R_\sigma(z)$ is not new, but is a result of Laptev and Weidl [26,28], for which we provide an independent method of proof for

$\sigma \geq 2$. They regard it as a version of the Berezin-Li-Yau inequality, to which it is directly related by a Legendre transform when $\sigma = 1$. See also [20] for a discussion of the various connections.

Proof. Since $R_\sigma(z) \geq (z_0 - \lambda_1)^\sigma_+$, for any $z_0 > \lambda_1$, it follows from Theorem 2.1 that for all $z \geq z_0$,

$$R_\sigma(z) \geq (z_0 - \lambda_1)^\sigma_+ \left(\frac{z}{z_0}\right)^{\sigma + \frac{d}{2}}.$$

By a straightforward calculation, the coefficient of $z^{\sigma + \frac{d}{2}}$ is maximized when $z_0 = \left(1 + \frac{2\sigma}{d}\right) \lambda_1$, and the lower bound that results is the left side of (2.11).

As for the other inequality, note, following [26,28], that the Weyl law implies that

$$\frac{R_\sigma(z)}{z^{\sigma + \frac{d}{2}}} \to L^{cl}_{\sigma,d} |\Omega|$$

as $z \to \infty$. Since $\frac{R_\sigma(z)}{z^{\sigma + \frac{d}{2}}}$ is a nondecreasing function, it is less than this limit for all finite z. □

Remark 2.5 It is also possible to prove the upper bound in (2.11) by invoking the difference inequality (2.4) directly. Note that (2.4) can be rewritten in the form

$$\frac{R_{\sigma-1}(z)}{L^{cl}_{\sigma-1,d} z^{\sigma-1+\frac{d}{2}}} \geq \frac{R_\sigma(z)}{L^{cl}_{\sigma,d} z^{\sigma+\frac{d}{2}}}, \tag{2.13}$$

by virtue of the fact that

$$L^{cl}_{\sigma-1,d} = \left(1 + \frac{d}{2\sigma}\right) L^{cl}_{\sigma,d}.$$

Ineq. (2.13) is a version of a monotonicity principle of Aizenman and Lieb [1] at the level of the ratios rather than their suprema. The proof is then complete owing to the monotonicity principle for $R_\sigma(z)/z^{\sigma+d/2}$ and the Weyl asymptotic law as applied to $R_{\sigma-1}(z)/z^{\sigma-1+d/2}$, i.e., for $z \geq z_0 > 0$

$$\frac{R_{\sigma-1}(z)}{L^{cl}_{\sigma-1,d} z^{\sigma-1+\frac{d}{2}}} \geq \frac{R_\sigma(z)}{L^{cl}_{\sigma,d} z^{\sigma+\frac{d}{2}}} \geq \frac{R_\sigma(z_0)}{L^{cl}_{\sigma,d} z_0^{\sigma+\frac{d}{2}}}.$$

Sending $z \to \infty$ results in the Berezin-Li-Yau inequality for $\sigma \geq 2$ once more.

Actually, the monotonicity principle described in Theorem 2.1 implies that for sufficiently large z_0, the lower bound on $R_\sigma(z)/z^{\sigma+\frac{d}{2}}$ is arbitrarily close to the upper bound. That is, given any domain Ω and any $\epsilon > 0$, there exists a large but finite z_ϵ such that for $z \geq z_\epsilon$,

$$R_\sigma \geq \left(L^{cl}_{\sigma,d} - \epsilon\right) |\Omega| z^{\sigma+\frac{d}{2}}.$$

In this estimate, however, z_ϵ is not independent of the shape of Ω.

Arguments nearly identical to those adduced for Corollary 2.3 lead to lower bounds on R_σ when $\sigma < 2$, but they are not of Weyl-type due to the different exponent of z in Theorem 2.1. To remedy this deficiency, we call on (2.2) and (2.4):

Corollary 2.6 *For $1 \le \sigma < 2$ and $z \ge \left(1 + \frac{2\sigma+2}{d}\right)\lambda_1$,*

$$R_\sigma(z) \ge \frac{(2\sigma+2)^\sigma d^{\frac{d}{2}}}{(d+2\sigma+2)^{\sigma+\frac{d}{2}}} \lambda_1^{-\frac{d}{2}} z^{\sigma+\frac{d}{2}}, \tag{2.14}$$

and for $0 \le \sigma < 1$, $z \ge \left(1 + \frac{2\sigma+4}{d}\right)\lambda_1$,

$$R_\sigma(z) \ge \frac{\left(1 + \frac{d}{4}\right)(2\sigma+4)^{\sigma+1} d^{\frac{d}{2}}}{(d+2\sigma+4)^{\sigma+1+\frac{d}{2}}} \lambda_1^{-\frac{d}{2}} z^{\sigma+\frac{d}{2}}. \tag{2.15}$$

For convenience we collect the most important cases $\sigma = 0, 1$, which, by Theorem 2.1, simplify to:

$$R_1(z) \ge \left(1 + \frac{d}{4}\right)\frac{1}{z} R_2(z) \ge \frac{4 \, d^{\frac{d}{2}}}{(d+4)^{1+\frac{d}{2}}} \lambda_1^{-\frac{d}{2}} z^{1+\frac{d}{2}}, \tag{2.16}$$

and,

$$\mathcal{N}(z) = R_0(z) \ge \left(1 + \frac{d}{4}\right)^2 \frac{1}{z^2} R_2(z) \ge \left(\frac{z}{\left(1 + \frac{4}{d}\right)\lambda_1}\right)^{\frac{d}{2}}. \tag{2.17}$$

Remark 2.7 The Cheng-Yang bound (1.7) is a simple corollary of (2.17); for the proof let z approach λ_{k+1} and collect terms. These inequalities and the Cheng-Yang bound will be improved and generalized below. Furthermore, the lower bounds provided by (2.11) and (2.14) compete with the bound

$$R_\sigma(z) \ge H_d^{-1} \lambda_1^{-d/2} \frac{\Gamma(1+\sigma)\Gamma(1+d/2)}{\Gamma(1+\sigma+d/2)} \ (z - \lambda_1)_+^{\sigma+d/2}, \tag{2.18}$$

valid for $\sigma \ge 1$, where H_d is defined by (1.6) above. The case $\sigma = 1$ of (2.18) can be found in [22] and is in fact hidden in earlier work of Laptev [26]. It is the Legendre transform of this inequality that provides the bound (1.5) which competes with Cheng and Yang's (1.7). Additional discussion and comparisons appear in Section 4 (see also [20]).

One option for generalizing Inequalities (2.17) and (1.7) is to compare with parts of the spectrum lying above λ_1. To avoid complications, we shall focus on inequalities involving $R_2(z)$. Just as $\overline{\lambda_j}$ denotes the mean of the eigenvalues

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λ_ℓ for $\ell \le j$, define

$$\overline{\lambda_j^2} := \frac{1}{j} \sum_{\ell \le j} \lambda_\ell^2.$$

Then setting $j = [z]$ (greatest integer $\le z$), we find that

$$R_2(z) = [z] \left(z^2 - 2z\overline{\lambda_{[z]}} + \overline{\lambda_{[z]}^2} \right). \tag{2.19}$$

For any integer j and all $z \ge j$,

$$R_2(z) \ge Q(z, j) := j \left(z^2 - 2z\overline{\lambda_j} + \overline{\lambda_j^2} \right).$$

Using Theorem 2.1, for $z \ge z_j \ge \lambda_j$,

$$R_2(z) \ge Q(z_j, j) \left(\frac{z}{z_j} \right)^{2 + \frac{d}{2}}. \tag{2.20}$$

A good – simple but not optimized – choice is $z_j = \left(1 + \frac{4}{d} \right) \overline{\lambda_j}$. Observe that because of the Cauchy-Schwarz inequality, $\overline{\lambda_j}^2 \le \overline{\lambda_j^2}$, so

$$Q(z, j) = j \left(\left(z - \overline{\lambda_j} \right)^2 + \overline{\lambda_j^2} - \overline{\lambda_j}^2 \right)$$

$$\ge j \left(z - \overline{\lambda_j} \right)^2. \tag{2.21}$$

This establishes the following.

Corollary 2.8 *Suppose that* $z \ge \left(1 + \frac{4}{d} \right) \overline{\lambda_j}$. *Then*

$$R_2(z) \ge \frac{j z^{2 + \frac{d}{2}}}{\left(1 + \frac{d}{4} \right)^2 \left(\left(1 + \frac{4}{d} \right) \overline{\lambda_j} \right)^{\frac{d}{2}}}, \tag{2.22}$$

and therefore,

$$R_1(z) \ge \frac{j z^{1 + \frac{d}{2}}}{\left(1 + \frac{d}{4} \right) \left(\left(1 + \frac{4}{d} \right) \overline{\lambda_j} \right)^{\frac{d}{2}}} \tag{2.23}$$

and

$$\mathcal{N}(z) \ge j \left(\frac{z}{\left(1 + \frac{4}{d} \right) \overline{\lambda_j}} \right)^{\frac{d}{2}}. \tag{2.24}$$

Moreover, for all $k \ge j \ge 1$,

$$\lambda_{k+1} / \overline{\lambda_j} \le \left(1 + \frac{4}{d} \right) \left(\frac{k}{j} \right)^{\frac{2}{d}}. \tag{2.25}$$

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Proof. The first statement comes from substituting $z = z_j = \left(1 + \frac{4}{d}\right)\overline{\lambda_j}$ into Eqs. (2.20) and (2.21) and simplifying. The next two statements result from substituting the first statement into (2.16) and (2.17).

The final statement (2.25) is automatic if $\lambda_{k+1} \leq \left(1 + \frac{4}{d}\right)\overline{\lambda_j}$. Suppose to the contrary that $\lambda_{k+1} > \left(1 + \frac{4}{d}\right)\overline{\lambda_j}$. As z increases to λ_{k+1}, (2.24) becomes valid and in the limit reads

$$k \geq j \left(\frac{\lambda_{k+1}}{\left(1 + \frac{4}{d}\right)\overline{\lambda_j}}\right)^{\frac{d}{2}},$$

which when solved for $\lambda_{k+1}/\overline{\lambda_j}$ yields the claim. $\qquad\square$

The case $j = k$ reproduces a straightforward and well-known simplification of Yang's inequality [37], *viz.*,

$$\lambda_{k+1} \leq \left(1 + \frac{4}{d}\right)\overline{\lambda_k}. \tag{2.26}$$

The case $j = 1$ reduces to the Cheng-Yang bound (1.7). The other cases are new.

We turn now to the relation of Riesz means having values of σ not necessarily differing by integers as in Theorem 2.1. Because the upper and lower bounds in (2.11) obey the same power law, all Riesz means for $\sigma \geq 2$ are comparable in the sense that for any pair $\sigma_{1,2} \geq 2$, and all z larger than a certain value,

$$c_\ell \leq \frac{R_{\sigma_1}(z)z^{-\sigma_1}}{R_{\sigma_2}(z)z^{-\sigma_2}} \leq c_u,$$

for explicit non-zero quantities $c_{\ell,u}$ depending only on $\sigma_{1,2}$, the dimension d, and the dimensionless quantity $|\Omega|\lambda_1^{\frac{d}{2}}$. Moreover, the subsequent corollary, together with the Laptev-Weidl version of Berezin-Li-Yau [28] allows the same claim to be made for $\sigma_{1,2} \geq 1$, with the upper bound holding even for $\sigma_2 \geq 0$.

Other constraints on ratios of Riesz means derive from Hölder's inequality, by which $R_\sigma(z)$ is a log-convex function of σ. That is, for $\sigma_0 \leq \sigma_1 \leq \sigma_2$, if t is chosen so that $\sigma_1 = t\sigma_0 + (1 - t)\sigma_2$, Hölder's inequality applied to the definition (2.1) directly produces:

$$R_{\sigma_1}(z) \leq R_{\sigma_0}(z)^t R_{\sigma_2}(z)^{1-t}. \tag{2.27}$$

Choosing $\sigma_0 = 0, \sigma_1 = \sigma - 1, \sigma_2 = \sigma$, and $t = \frac{1}{\sigma}$, we get

$$R_{\sigma-1}(z) \leq (R_\sigma(z))^{\frac{\sigma-1}{\sigma}} (R_0(z))^{\frac{1}{\sigma}},$$

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or, equivalently,

$$\mathcal{N}(z) \geq \frac{(R_{\sigma-1}(z))^{\sigma}}{(R_{\sigma}(z))^{\sigma-1}}. \tag{2.28}$$

Along with (2.4), we obtain, for all z and $\sigma \geq 2$,

$$\mathcal{N}(z) \geq \left(\frac{d+2\sigma}{2\sigma}\right)^{\sigma} z^{-\sigma} R_{\sigma}(z). \tag{2.29}$$

From this stage lower bounds on $\mathcal{N}$, and consequently upper bounds on λ_{k+1}, can be derived for all $\sigma \geq 2$, by optimizing coefficients as before to eliminate R_{σ}. The latter, however, are not found to improve upon the case $\sigma = 2$, which corresponds to (2.17) and (2.24).

3   Weyl-type estimates of means of eigenvalues

The Legendre transform is an effective tool for converting bounds on $R_{\sigma}(z)$ into bounds on the spectrum, as has been realized previously, e.g., in [29]. We use it in this section to obtain some improvements on parts of the preceding section, with a focus on the averages $\overline{\lambda_k}$.

Recall that if $f(z)$ is a convex function on $\mathbb{R}^+$ that is superlinear in z as $z \to +\infty$, its *Legendre transform*

$$\mathcal{L}[f](w) := \sup_{z} \{wz - f(z)\} \tag{3.1}$$

is likewise a superlinear convex function. Moreover, for each w, the supremum in this formula is attained at some finite value of z. (A concise treatment of the Legendre transform may be found for example in chapter 3 of [17]. The Legendre transform on $\mathbb{R}^+$ can be understood in terms of the more widely treated Legendre transform on $\mathbb{R}$ by restricting to even functions.) We also note that $f(z) \geq g(z)$ for all $z \implies \mathcal{L}[f](w) \leq \mathcal{L}[g](w)$ for all w, and proceed to calculate the Legendre transform of (2.23).

First we make a remark about the restriction on the range of values of z for which Ineq. (2.23) is valid, *viz.*, $z \geq z_j := \left(1 + \frac{4}{d}\right) \overline{\lambda_j}$. We can imagine redefining the function on the right for $z \leq z_j$ in some unimportant way, though preserving convexity, so that it provides a lower bound to the left side for all $z \geq 0$. While this extended definition contains no information, it ensures that both sides of Ineq. (2.16) are now defined on a standard interval guaranteeing that the Legendre transform has the properties cited above. Since the maximizing value of z in the definition (3.1) is a nondecreasing function of w, it follows that for w sufficiently large, the maximizing z exceeds z_j. This permits us simply to transform both sides of Ineq. (2.16) to obtain a dual

inequality that is valid when $w \geq w_j$, for some w_j, the value of which we shall estimate *post facto*.

The Legendre transforms of the two sides of Ineq. (2.23) are straightforward: The maximizing value of z in the Legendre transform of R_1 is attained at one of the critical values, which, because R_1 is piecewise linear, means that $z_{cr} = \lambda_{k+1}$, and it is easy to check that $k = [w]$. Substitution reveals

$$\mathcal{L}\left[R_1\right](w) = (w - [w]) \lambda_{[w]+1} + [w] \overline{\lambda_{[w]}}.$$

Together with the standard Legendre transform of the right side of (2.23), we obtain

$$(w - [w]) \lambda_{[w]+1} + [w] \overline{\lambda_{[w]}} \leq \frac{2}{j^{\frac{2}{d}}} \left(\frac{1 + \frac{d}{4}}{1 + \frac{d}{2}}\right)^{1+\frac{2}{d}} \overline{\lambda_j}\, w^{1+\frac{2}{d}} \qquad (3.2)$$

for certain values of w and j, the determination of which we now consider. In Corollary 2.8 it is supposed that $z \geq \left(1 + \frac{4}{d}\right)\overline{\lambda_j}$, and we recall that that there is a monotonic relationship between w and the maximizing z_* in the calculation of the Legendre transform of the right side of (2.23). (By the maximizing z_* for a Legendre transform we mean the value such that $\mathcal{L}[f](w) = wz_* - f(z_*)$.) By an elementary calculation,

$$w = j \left(\frac{1 + \frac{d}{2}}{1 + \frac{d}{4}}\right) \left(\frac{z_*}{\left(1 + \frac{4}{d}\right)\overline{\lambda_j}}\right)^{\frac{d}{2}}, \qquad (3.3)$$

so it follows that if w is restricted to values $\geq j\frac{1+\frac{d}{2}}{1+\frac{d}{4}}$, then Ineq. (3.2) is valid. Meanwhile, for any w we can always find an integer k such that on the left side of (3.2), $k - 1 \leq w < k$. If $k > j\frac{1+\frac{d}{2}}{1+\frac{d}{4}}$ and if we let w approach k from below, we obtain from (3.2)

$$\lambda_k + (k-1)\overline{\lambda_{k-1}} \leq \frac{2}{j^{\frac{2}{d}}} \left(\frac{1 + \frac{d}{4}}{1 + \frac{d}{2}}\right)^{1+\frac{2}{d}} \overline{\lambda_j} k^{1+\frac{2}{d}}.$$

The left side of this equation is the sum of the eigenvalues λ_1 through λ_k, and the calculation can thus be encapsulated as

Corollary 3.1 *For $k \geq j\frac{1+\frac{d}{2}}{1+\frac{d}{4}}$, the means of the eigenvalues of the Dirichlet Laplacian satisfy a universal Weyl-type bound,*

$$\overline{\lambda_k}/\overline{\lambda_j} \leq 2 \left(\frac{1 + \frac{d}{4}}{1 + \frac{d}{2}}\right)^{1+\frac{2}{d}} \left(\frac{k}{j}\right)^{\frac{2}{d}}. \qquad (3.4)$$

Note that for small values of j the restriction on k is fulfilled simply for

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$k > j$. We are aware of only one previous bound somewhat comparable to Corollary 3.1, namely in the case $k = d + 1, j = 1$, for which Ashbaugh and Benguria [5] have shown that

$$\overline{\lambda_{d+1}}/\lambda_1 \leq \frac{d+5}{d+1}.$$

This is stronger than (3.4) when $k = d + 1, j = 1$. Hence we can set $j = d + 1$ and multiply the two bounds, to obtain:

Corollary 3.2 *For $k \geq \frac{(d+1)\,(1+\frac{d}{2})}{1+\frac{d}{4}}$,*

$$\overline{\lambda_k}/\lambda_1 \leq \frac{d+5}{2^{\frac{2}{d}}} \left(\frac{(d+4)}{(d+1)(d+2)} \right)^{1+\frac{2}{d}} k^{\frac{2}{d}}. \tag{3.5}$$

We remark that together with (2.25), setting $j = k$, this implies

$$\lambda_{k+1}/\lambda_1 \leq \frac{(d+4)^{2+\frac{2}{d}}(d+5)}{2^{\frac{2}{d}}d(d+1)^{1+\frac{2}{d}}(d+2)^{1+\frac{2}{d}}} \, k^{\frac{2}{d}}. \tag{3.6}$$

Finally, we note that any bound in terms of $\overline{\lambda_k}$, whether from above or below, implies a bound, with $\overline{\lambda_k}$ replaced by the square root of $\overline{\lambda_k^2}$ with an additional factor depending only on dimension, since

$$\overline{\lambda_k}^2 \leq \overline{\lambda_k^2} \leq \frac{\left(1 + \frac{2}{d}\right)^2}{1 + \frac{4}{d}} \overline{\lambda_k}^2. \tag{3.7}$$

The first of these inequalities is Cauchy-Schwarz, and the second is from [21], Proposition 6 (ii) (set the parameter σ occurring there to 1). The bounds implied by the considerations above and (3.7) for the root-mean-square of the eigenvalues are of Weyl-type. Moreover, similar statements can be made about other eigenvalue moments

$$\mathcal{M}_\sigma := \mathcal{M}_\sigma(\lambda_1, \lambda_2, \ldots, \lambda_k) := \left(\frac{1}{k} \sum_{\ell \leq k} \lambda_\ell^\sigma \right)^{1/\sigma}$$

when $0 < \sigma \leq 2$, as well as for the geometric and harmonic means of the eigenvalues. In analogy to the situation of the Riesz means, this is a consequence of Hölder's inequality (see also [19]), which implies

$$\mathcal{M}_\sigma < \mathcal{M}_\mu \quad \text{for } \sigma < \mu,$$

and the interpolation formula

$$\mathcal{M}_\sigma^\sigma < \left(\mathcal{M}_\mu^\mu\right)^{\frac{\tau-\sigma}{\tau-\mu}} \left(\mathcal{M}_\tau^\tau\right)^{\frac{\sigma-\mu}{\tau-\mu}} \qquad \text{for } \mu < \sigma < \tau.$$

For $0 < \sigma < 1$, we specialize to the case

$$\mathcal{M}_\sigma < \mathcal{M}_1 = \overline{\lambda_k}.$$

For $1 < \sigma < 2$, Ineq. (3.7) can be combined with the interpolation formula for $\mu = 1$, $\tau = 2$ (Note that $\mathcal{M}_2 = \sqrt{\overline{\lambda_k^2}}$.) The claims for geometric and harmonic means are then clear by virtue of the classical inequalities relating these means to the arithmetic mean.

4   Comparison of universal bounds

In this section we compare our universal eigenvalue bounds with those that have appeared earlier. We consider bounds for $\overline{\lambda_k}$ as well as for λ_{k+1}. Setting $j = 1$ in (3.4), for $k \geq 2$, we obtain

$$\frac{\overline{\lambda_k}}{\lambda_1} \leq 2 \left(\frac{1 + \frac{d}{4}}{1 + \frac{d}{2}}\right)^{1 + \frac{2}{d}} k^{\frac{2}{d}}. \tag{4.1}$$

We claim that (4.1) is sharper than (1.7). To see this, set $j = 0, 1, \ldots k - 1$ in (1.7) and sum, to get

$$\sum_{j=0}^{k-1} \frac{\lambda_{j+1}}{\lambda_1} \leq \left(1 + \frac{4}{d}\right) \sum_{j=0}^{k-1} j^{\frac{2}{d}} \leq \left(1 + \frac{4}{d}\right) \frac{k^{1+\frac{2}{d}}}{1 + \frac{2}{d}},$$

where we have simplified the expression by regarding the middle sum as a left Riemann sum, following [22]. After simplification, the Cheng-Yang bound (1.7) implies

$$\frac{\overline{\lambda_k}}{\lambda_1} \leq \frac{d+4}{d+2} k^{\frac{2}{d}}. \tag{4.2}$$

The coefficients of (4.2) and (4.1) are plotted against the dimension in Fig. 1. Clearly,

$$1 < 2 \left(\frac{1 + \frac{d}{4}}{1 + \frac{d}{2}}\right)^{1 + \frac{2}{d}} < \frac{d+4}{d+2}.$$

The averaged bounds are also plotted in Fig. 2 against k (for $d = 4$) along with the bound of Hermi (1.5) and with (4.2) and (4.1), and are seen to improve the earlier results. The averaged versions of the Ashbaugh-Benguria bound (1.3)
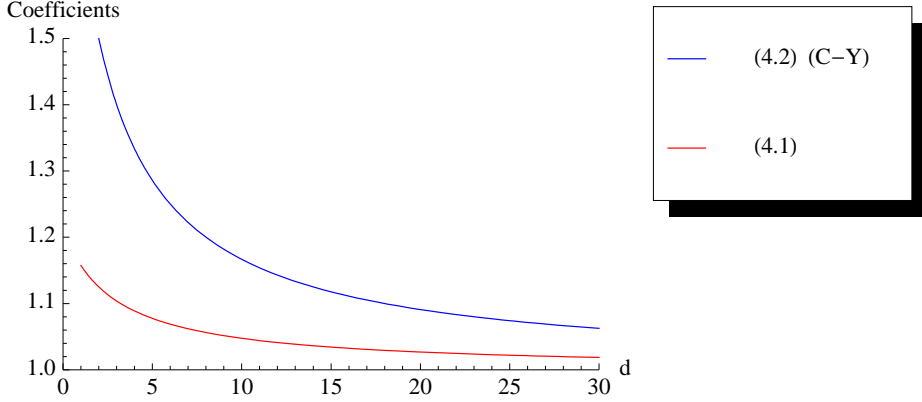
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Fig. 1. (4.1) vs (4.2) as a function of d .

Table 1

Bound for $\dfrac{\overline{\lambda}_{127}}{\lambda_1}$ as a function of the dimension d.

d	(4.1)	(4.2)	(1.5)	$\overline{AB}$	$\overline{Weyl}$
2	142.875	190.5	163.962	339.852	43.9204
3	27.8886	35.3723	32.5332	89.974	8.9804
4	12.2686	15.0259	14.7695	40.2459	4.0937
5	7.48017	8.92619	9.34082	23.3009	2.56781
6	5.37202	6.28316	6.95603	15.646	1.88786
7	4.23768	4.87795	5.67474	11.5391	1.51906

and that of the Weyl expression in (1.10) are also illustrated along with (4.2) and (4.1) in Fig. 2. They are obtained by dividing the expressions in those equations by $1 + \frac{2}{d}$ (i.e., performing the integration in k). This comparison between the various bounds is displayed numerically for $\overline{\lambda}_{127}/\lambda_1$ for various dimensions in Table 1.

Notice again the competition between the various bounds with (4.1) always providing the best constant. One can also combine (4.1) and (2.26). The bounds that ensue, however, were found not to fare better than (1.7). Note that we improve by almost 25 percent going from (1.7) to (4.1) when $d = 2$; see Table 2.

Similar comparisons can be made for $\overline{\lambda}_k$ with $k \geq d + 1$, setting (3.5) beside the bound (1.9) found in [14]. Again, on average, (3.5) is an improvement. The comparison is displayed in Table 3 when $k = \left[\frac{(d+1)\,(1+\frac{d}{2})}{1+\frac{d}{4}} \right] + 1$. The columns in Table 3 represent the ratios between the expressions found in (3.5) and (1.9) and the averaged version of (1.10).
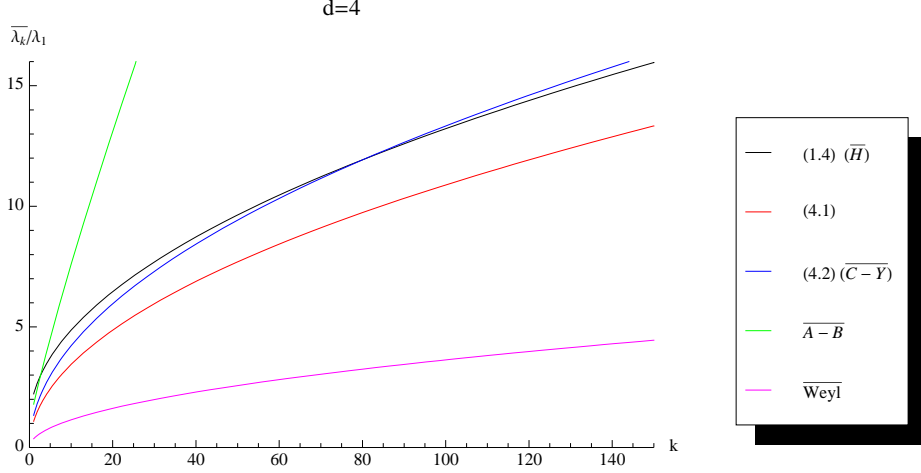
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Fig. 2. Bound for $\overline{\lambda}_k/\lambda_1$ as a function of k ($d = 4$).

Table 2

Comparison for coefficients (4.1) and (4.2) as a function of the dimension d.

d	$(4.1)/(\overline{Weyl})$	$(4.2)/(\overline{Weyl})$
2	3.250304	4.33739
3	3.10528	3.93884
4	2.99694	3.67049
5	2.91306	3.47619
6	2.84556	3.32818
7	2.78967	3.21116

Table 3

Bound for (3.5) and (1.9) as a function of the dimension d, for $k = \left[\frac{(d+1)\left(1+\frac{d}{2}\right)}{1+\frac{d}{4}}\right] + 1$.

d	$(3.5)/(\overline{Weyl})$	$(1.9)/(\overline{Weyl})$
2	2.53014	3.08587
3	2.46466	2.92435
4	2.41249	2.80499
5	2.37103	2.71385
6	2.33756	2.64210
7	2.31003	2.58414

We conclude by noting, as discussed above, that for low-lying eigenvalues one cannot expect to improve on the bound (1.3) of Ashbaugh-Benguria . This is illustrated in Fig. 3.
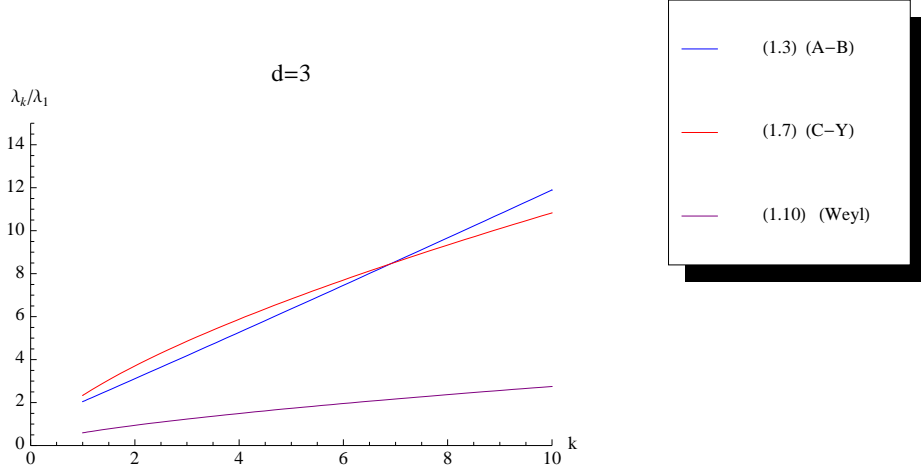
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Fig. 3. (1.3) vs (1.7) as a function of k ($d = 3$).

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